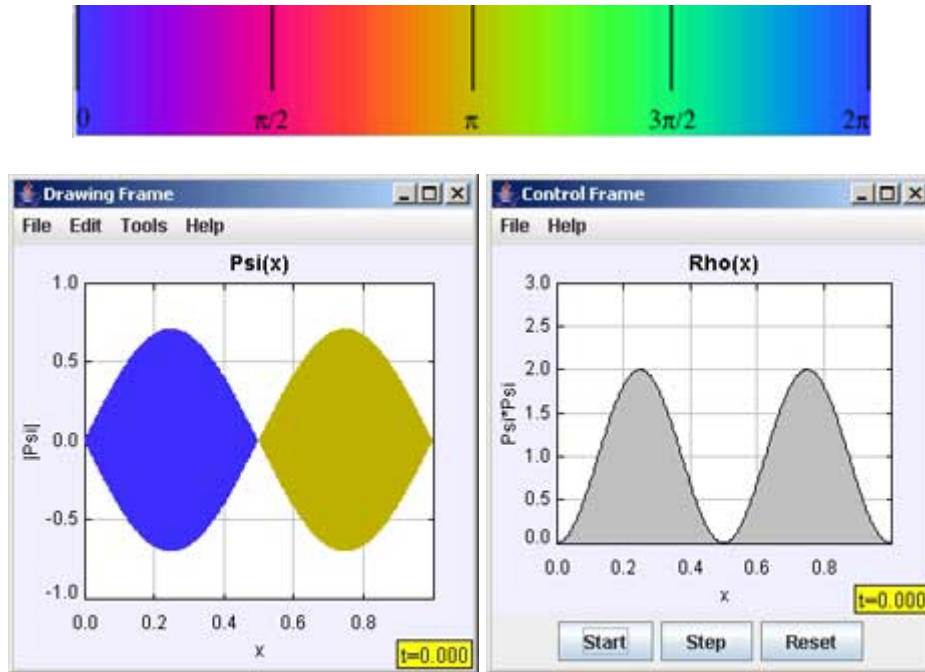


To access these exercises, double-click the “osp\_qm\_superposition.jar” file. When the jar opens, select the “Quantum Mechanics Worksheets” button at the bottom of the splash screen. Navigate to the “Worksheet #2” folder where you will find the interactive curricular materials that go with these exercises.

## OSP QM Worksheet # 2: Energy Eigenstates: Time Evolution



Time-dependent quantum-mechanical wave functions are inherently complex (having real and imaginary components) due to the time evolution governed by the Schrödinger equation, which in one-dimensional position space is

$$[-(\hbar^2/2m)\partial^2/\partial x^2 + V(x)] \psi(x,t) = i\hbar(\partial/\partial t) \psi(x,t) .$$

The standard way to visualize the wave functions that solve this equation is to either consider just the real part or consider the probability density, approaches that discard all phase information. We use color-as-phase representation of the wave function to display this information in a meaningful way. Such a time-dependent state in the infinite square well experiences so-called quantum-mechanical revivals, in which an initially localized wave packet reforms a definite time later.

### Energy Eigenstate Shape

Begin by double-clicking the **Eigenstate Shape Overview** node (the green arrow). Note the red and blue components and the state's shape.

Wave functions in quantum mechanics are inherently complex (having real and imaginary components) due to the time evolution governed by the Schrödinger equation, which in one-dimensional position space is

$$[-(\hbar^2/2m)\partial^2/\partial x^2 + V(x)] \psi(x,t) = i\hbar(\partial/\partial t) \psi(x,t) .$$

The solutions to this equation are called wave functions,  $\psi(x,t)$ . When one separates out the time dependence in the Schrödinger equation, one is left with the time-independent Schrödinger equation.

$$[-(\hbar^2/2m)d^2/dx^2 + V(x)] \psi_n(x) = E_n \psi_n(x) .$$

This is where much of quantum mechanics starts once one specifies the potential energy function,  $V(x)$ . The time-independent Schrödinger equation is also called an energy eigenvalue equation as the solutions to this equation yield states of definite energy,  $E_n$ , called energy eigenstates. We can determine the energy eigenstate shape by rewriting the time-independent Schrödinger equation as

$$\{ d^2/dx^2 + (2m/\hbar^2)[ E_n - V(x) ] \} \psi_n(x) = 0 .$$

The shape of the energy eigenstate is dependent on the local value of  $E_n - V(x)$ .<sup>1</sup> In regions where  $E_n - V(x) > 0$  the energy eigenstate curves toward the axis and exhibits oscillatory behavior and in regions where  $E_n - V(x) < 0$  the energy eigenstate curves away from the axis and exhibits exponential behavior. The larger the difference between  $E_n$  and  $V(x)$ , the stronger the behavior becomes. The exception occurs in regions where  $E_n - V(x) = 0$ , where the energy eigenstate does not curve at all, and exhibits linear behavior.<sup>2</sup>

<sup>1</sup>See for example the classic paper, A. P. French and E. F. Taylor, "Qualitative plots of bound state wave functions," Am. J. Phys. **39**, 961-962 (1971).

<sup>2</sup>See for example, L. P. Gilbert, *et. al.*, "More on the Asymmetric Infinite Square Well: Energy Eigenstates with Zero-curvature," Eur. J. Phys. **26**, 815-825 (2005).

## Energy Eigenstate Time Evolution

Begin by double-clicking the **Time Evolution Overview** node (the green arrow). Note initial colors and shape of the state depicted. Press the Start button, and watch the state evolve in time. Note what changes and what stays the same.

Wave functions in quantum mechanics are inherently complex (having real and imaginary components) due to the time evolution governed by the Schrödinger equation, which in one-dimensional position space is

$$[-(\hbar^2/2m)\partial^2/\partial x^2 + V(x)] \psi(x,t) = i\hbar(\partial/\partial t) \psi(x,t) .$$

The solutions to this equation are called wave functions  $\psi(x,t)$ . When one separates out the time dependence in the Schrödinger equation, one is left with the time-independent Schrödinger equation.

$$[-(\hbar^2/2m)d^2/dx^2 + V(x)] \psi_n(x) = E_n \psi_n(x) .$$

This is where much of quantum mechanics starts once one specifies the potential energy function,  $V(x)$ . The time-independent Schrödinger equation is also called an energy eigenvalue equation as the solutions to this equation yield states of definite energy,  $E_n$ , called energy eigenstates. Energy eigenstates are special in a number of ways. First, when their energy is measured, the result is a definite energy, always. When one determines the time dependence of energy eigenstates, one finds that they evolve in time according to

$$\psi_n(x,t) = \exp(-iE_n t/\hbar) \psi_n(x),$$

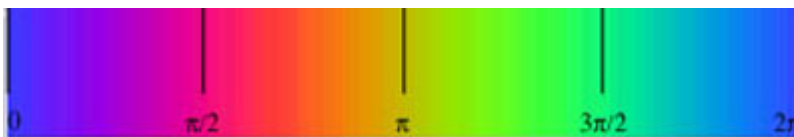
which satisfies the Schrödinger equation. There are two ways to represent complex functions such as this. One way is to split the function into its real and imaginary parts

$$\text{Real part: } \cos(-iE_n t/\hbar) \psi_n(x) \text{ and Imaginary part: } \sin(-iE_n t/\hbar) \psi_n(x),$$

the other way is to represent the function in amplitude-and-phase representation

$$\psi_n(x,t) = \exp(-iE_n t/\hbar) [ \psi_n^*(x) \psi_n(x) ]^{1/2}.$$

These animations use amplitude-and-phase representation by showing the amplitude as the height on the plot and the color as the phase or phase angle. The chart below converts color to phase (angle) such that blue is positive and real, gold is negative and real, pink is positive and imaginary, and green is negative and imaginary.

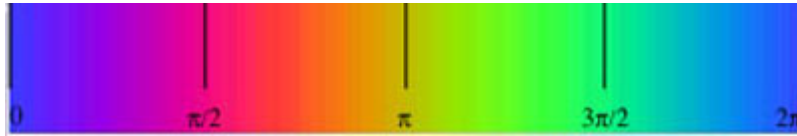


Note that the time dependence provides an overall, time-dependent phase factor to the energy eigenstate. Is this time dependence important? An important construct to answer this question is the probability density. The probability density (shown here in position space) for an energy eigenstate is  $|\psi_n(x,t)|^2 = \psi_n^*(x,t) \psi_n(x,t) = \psi_n^*(x) \psi_n(x)$  because the time dependence in the exponentials cancel. Therefore energy eigenstates yield time-independent probability densities. As a consequence, they also yield time-independent expectation values such as  $\langle x \rangle$ ,  $\langle p \rangle$ , and  $\langle E \rangle$ . This time-independence property is why these states are often also called "stationary states."

(Questions on next page)

## Infinite Square Well Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical infinite square well: a well with zero potential energy between  $x = 0$  and  $x = L$  and infinite walls at  $x = 0$  and  $x = L$ . Here  $L = 1$ .



**Note: animations use color-as-phase representation. The chart above converts color to phase (angle) such that blue is positive and real, gold is negative and real, pink is positive and imaginary, and green is negative and imaginary.**

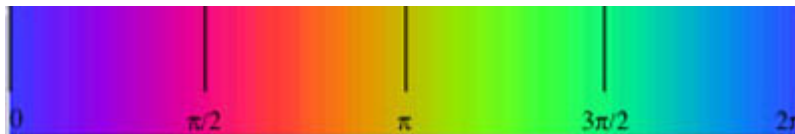
1. Write the energy function for the infinite square well. Here,  $\hbar = 2m = 1$  and  $L = 1$ . What is the energy function in this special case? In going from the ground state ( $n = 1$ ) to the first-excited state ( $n = 2$ ), the energy changes by what factor? In going from the ground state ( $n = 1$ ) to the second excited state ( $n = 3$ ), the energy changes by what factor?
  
2. Load each state by double-clicking its green arrow. Describe how the shape of the energy eigenstates (in both position and momentum space) change as  $n$  increases ( $t = 0$ ). Explain why these states should change in this way as  $n$  increases. Based on this answer, describe and sketch the  $n = 3$  and  $n = 4$  energy eigenstates (which are not depicted).

3. Watch (play using the Start button) the  $n = 1$  energy eigenstate in both representations (position space with its probability density and momentum space). How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?
  
4. Watch (play) the  $n = 2$  energy eigenstate in both representations. How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?
  
5. Watch (play) the  $n = 5$  and  $n = 6$  energy eigenstates in both representations. How long do these states take to time evolve back to their original orientations? As a consequence, what are the energy eigenvalues of these states?
  
6. How long would it take a general energy eigenstate, therefore, to time evolve back to its original orientation? Base your results on your answers to Questions 3 - 5. Hint: the inverse of the time is proportional to what property of this energy eigenstate?

7. The expectation value of  $x$ ,  $\langle x \rangle$ , is essentially an average of  $x$  weighted by the probability density in position space. The expectation value of  $p$ ,  $\langle p \rangle$ , is essentially an average of  $p$  weighted by the probability density in momentum space. What are  $\langle x \rangle$  and  $\langle p \rangle$  for  $t = 0$  for these energy eigenstates?
8. Does the probability density,  $\langle x \rangle$ , or  $\langle p \rangle$  change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?
9. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability densities, and  $\langle x \rangle$  and  $\langle p \rangle$  to change over time for these superpositions? Why or why not?

## Harmonic Oscillator Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical harmonic oscillator: a well with  $V(x) = m\omega^2 x^2/2$ .



**Note: animations use color-as-phase representation. The chart above converts color to phase (angle) such that blue is positive and real, gold is negative and real, pink is positive and imaginary, and green is negative and imaginary.**

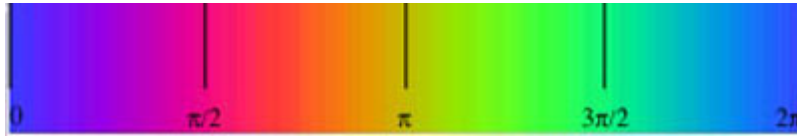
1. Write the energy function for the harmonic oscillator. Here,  $\hbar = 2m = 1$  and  $\omega = 1$ . What is the energy function in this special case? In going from the ground state ( $n = 0$ ) to the first-excited state ( $n = 1$ ), the energy changes by what factor? In going from the ground state ( $n = 0$ ) to the second excited state ( $n = 2$ ), the energy changes by what factor?
  
  
  
  
  
  
  
  
  
  
2. Load each state by double-clicking its green arrow. Describe how the shape of the energy eigenstates (in both position and momentum space) change as  $n$  increases ( $t = 0$ ). Explain why these states should change in this way as  $n$  increases. Based on this answer, describe and sketch the  $n = 2$  and  $n = 3$  energy eigenstates (which are not depicted).

3. Watch (play using the Start button) the  $n = 0$  energy eigenstate in both representations (position space with its probability density and momentum space). How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?
  
4. Watch (play) the  $n = 1$  energy eigenstate in both representations. How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?
  
5. Watch (play) the  $n = 4$  and  $n = 5$  energy eigenstates in both representations. How long do these states take to time evolve back to their original orientations? As a consequence, what are the energy eigenvalues of these states?
  
6. How long would it take a general energy eigenstate, therefore, to time evolve back to its original orientation? Base your results on your answers to Questions 3 - 5. Hint: the inverse of the time is proportional to what property of this energy eigenstate? How does this result compare to a similar result for the infinite square well energy eigenstates?

7. The expectation value of  $x$ ,  $\langle x \rangle$ , is essentially an average of  $x$  weighted by the probability density in position space. The expectation value of  $p$ ,  $\langle p \rangle$ , is essentially an average of  $p$  weighted by the probability density in momentum space. What are  $\langle x \rangle$  and  $\langle p \rangle$  for  $t = 0$  for these energy eigenstates?
8. Does the probability density,  $\langle x \rangle$ , or  $\langle p \rangle$  change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?
9. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability densities, and  $\langle x \rangle$  and  $\langle p \rangle$  to change over time for these superpositions? Why or why not?

## Asymmetric Infinite Square Well Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical asymmetric infinite square well: a well with zero potential energy between  $x = -L$  and  $x = 0$ , a constant potential energy,  $V_0$ , between  $x = 0$  and  $x = L$ , and infinite walls at  $x = -L$  and  $x = L$ . Here  $L = 1$ .



**Note: animations use color-as-phase representation. The chart above converts color to phase (angle) such that blue is positive and real, gold is negative and real, pink is positive and imaginary, and green is negative and imaginary.**

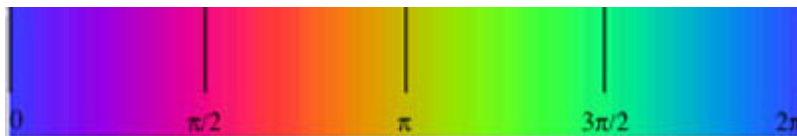
1. Load each state by double-clicking its green arrow. Describe how the shape of the energy eigenstates (in both position and momentum space) change as  $n$  increases ( $t = 0$ ). Explain why these states should change in this way as  $n$  increases. Based on this answer, describe and sketch the  $n = 3$  and  $n = 4$  energy eigenstates (which are not depicted).
  
2. Watch (play using the Start button) the  $n = 1$  energy eigenstate in both representations (position space with its probability density and momentum space). How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?

3. Watch (play) the  $n = 2$  energy eigenstate in both representations. How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?
  
4. Watch (play) the  $n = 5$  and  $n = 6$  energy eigenstates in both representations. How long do these states take to time evolve back to their original orientations? As a consequence, what are the energy eigenvalues of these states?
  
5. Can you predict how long would it take a general energy eigenstate, therefore, to time evolve back to its original orientation? Base your results on your answers to Questions 2 - 4.



## Ramped Infinite Square Well Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical ramped infinite well: a well with potential energy  $V(x) = \alpha x$  between  $x = -L$  and  $x = L$ , and infinite walls at  $x = -L$  and  $x = L$ . Here  $L = 1$ .



**Note: animations use color-as-phase representation. The chart above converts color to phase (angle) such that blue is positive and real, gold is negative and real, pink is positive and imaginary, and green is negative and imaginary.**

1. Load each state by double-clicking its green arrow. Describe how the shape of the energy eigenstates (in both position and momentum space) change as  $n$  increases ( $t = 0$ ). Explain why these states should change in this way as  $n$  increases. Based on this answer, describe and sketch the  $n = 3$  and  $n = 4$  energy eigenstates (which are not depicted).
  
  
  
  
  
  
  
  
  
  
2. Watch (play using the Start button) the  $n = 1$  energy eigenstate in both representations (position space with its probability density and momentum space). How long does this state take to time evolve back to its original orientation? As a consequence, what is the energy eigenvalue of this state?



6. The expectation value of  $x$ ,  $\langle x \rangle$ , is essentially an average of  $x$  weighted by the probability density in position space. The expectation value of  $p$ ,  $\langle p \rangle$ , is essentially an average of  $p$  weighted by the probability density in momentum space. What are  $\langle x \rangle$  and  $\langle p \rangle$  for  $t = 0$  for these energy eigenstates?
7. Does the probability density,  $\langle x \rangle$ , or  $\langle p \rangle$  change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?
8. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability densities, and  $\langle x \rangle$  and  $\langle p \rangle$  to change over time for these superpositions? Why or why not?