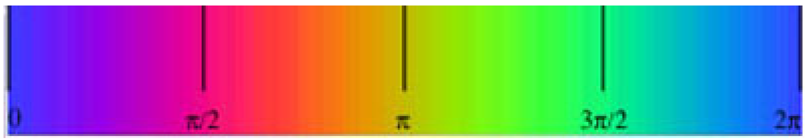
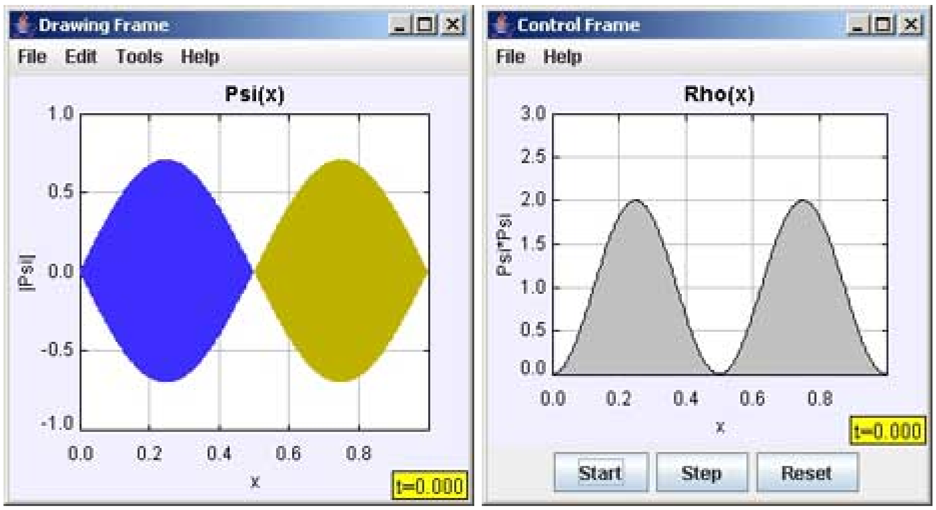
To access these exercises, double-click the "osp_qm_superposition.jar" file. When the jar opens, select the "Quantum Mechanics Worksheets" button at the bottom of the splash screen. Navigate to the "Worksheet #1" folder where you will find the interactive curricular materials that go with these exercises.

OSP QM Worksheet # 1: Energy Eigenstates: Shape





Time-dependent quantum-mechanical wave functions are inherently complex (having real and imaginary components) due to the time evolution governed by the Schrödinger equation, which in one-dimensional position space is

$$[-(\hbar^2/2m)\partial^2/\partial x^2 + V(x)]\ \psi(x,t) = i\hbar(\partial/\partial t)\ \psi(x,t)\ .$$

The standard way to visualize the wave functions that solve this equation is to either consider just the real part or consider the probability density, approaches that discard all phase information. We use color-as-phase representation of the wave function to display this information in a meaningful way. Such a time-dependent state in the infinite square well experiences so-called quantum-mechanical revivals, in which an initially localized wave packet reforms a definite time later.

Energy Eigenstate Shape

Begin by double-clicking the **Eigenstate Shape Overview** node (the green arrow). Note the red and blue components and the state's shape.

Wave functions in quantum mechanics are inherently complex (having real and imaginary components) due to the time evolution governed by the Schrödinger equation, which in one-dimensional position space is

$$[-(\hbar^2/2m)\partial^2/\partial x^2 + V(x)]\ \psi(x,t) = i\hbar(\partial/\partial t)\ \psi(x,t)\ .$$

The solutions to this equation are called wave functions, $\psi(x,t)$. When one separates out the time dependence in the Schrödinger equation, one is left with the time-independent Schrödinger equation.

$$[-(\hbar^2/2m)d^2/dx^2 + V(x)]\ \psi_n(x) = E_n\ \psi_n(x)\ .$$

This is where much of quantum mechanics starts once one specifies the potential energy function, $V(x)$. The time-independent Schrödinger equation is also called an energy eigenvalue equation as the solutions to this equation yield states of definite energy, E_n, called energy eigenstates. We can determine the energy eigenstate shape by rewriting the time-independent Schrödinger equation as

$$\{\ d^2/dx^2 + (2m/\hbar^2)[\ E_n - V(x)\ ]\ \}\ \psi_n(x) = 0\ .$$

The shape of the energy eigenstate is dependent on the local value of $E_n - V(x)$.[1] In regions where $E_n - V(x) > 0$ the energy eigenstate curves toward the axis and exhibits oscillatory behavior and in regions where $E_n - V(x) < 0$ the energy eigenstate curves away from the axis and exhibits exponential behavior. The larger the difference between E_n and $V(x)$, the stronger the behavior becomes. The exception occurs in regions where $E_n - V(x) = 0$, where the energy eigenstate does not curve at all, and exhibits linear behavior.[2]

[1]See for example the classic paper, A. P. French and E. F. Taylor, "Qualitative plots of bound state wave functions," Am. J. Phys. **39**, 961-962 (1971).

[2]See for example, L. P. Gilbert, *et. al.*, "More on the Asymmetric Infinite Square Well: Energy Eigenstates with Zero-curvature," Eur. J. Phys. **26**, 815-825 (2005).

(Questions on next page)

Infinite Square Well Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical infinite square well: a well with zero potential energy between $x = 0$ and $x = L$ and infinite walls at $x = 0$ and $x = L$. Here $L = 1$.

For Questions 1 - 6 load, but do not start, the simulations. In other words, do not press the Start button.

1. Write the energy function for the infinite square well. Here, $\hbar = 2m = 1$ and $L = 1$. What is the energy function in this special case? In going from the ground state ($n = 1$) to the first-excited state ($n = 2$), the energy changes by what factor? In going from the ground state ($n = 1$) to the second-excited state ($n = 3$), the energy changes by what factor?

2. Load each state by double-clicking its green arrow. Describe and sketch how the shape of the energy eigenstates (in both position and momentum space) change as n increases ($t = 0$). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Explain why these states should change in this way as n increases. Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

3. Based on the answer to Question 2, describe and sketch the $n = 3$ and $n = 4$ energy eigenstates at $t = 0$ (which are not depicted).

4. Describe how the shape of the energy eigenstates' (in both position and momentum space) associated probability densities change as *n* increases (*t* = 0). Explain why these densities should change in this way as *n* increases. Based on this answer, describe and sketch the *n* = 3 and *n* = 4 probability densities at *t* = 0 (which are not depicted).

5. The expectation value of *x*, <*x*>, is essentially an average of *x* weighted by the probability density in position space. The expectation value of *p*, <*p*>, is essentially an average of *p* weighted by the probability density in momentum space. What are <*x*> and <*p*> for *t* = 0 for these energy eigenstates?

6. Based on your answers to Questions 1 - 4 predict what will happen to the energy eigenstates as they evolve in time. Do this for both the position-space and momentum-space representations. Also predict how the probability densities will evolve in time.

7. Now watch the simulations again, this time watching them time evolve (press the Start button on the simulation). How do the energy eigenstates (in both position and momentum space) actually evolve in time? How do the probability densities evolve in time?

8. Does the probability density, <*x*>, or <*p*> change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?

9. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability density, and <x> and <p> to change over time for these superpositions? Why or why not?

Harmonic Oscillator Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical harmonic oscillator: a well with $V(x) = m\omega^2x^2/2$.

For Questions 1 - 6 load, but do not start, the simulations. In other words, do not press the Start button.

1. Write the energy function for the harmonic oscillator. Here, $\hbar = 2m = 1$ and $\omega = 1$. What is the energy function in this special case? In going from the ground state ($n = 0$) to the first-excited state ($n = 1$), the energy changes by what factor? In going from the ground state ($n = 0$) to the second-excited state ($n = 2$), the energy changes by what factor?

2. Load each state by double-clicking its green arrow. Describe and sketch how the shape of the energy eigenstates (in both position and momentum space) change as n increases ($t = 0$). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Explain why these states should change in this way as n increases. Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

3. Based on the answer to Question 2, describe and sketch the $n = 2$ and $n = 3$ energy eigenstates at $t = 0$ (which are not depicted).

4. Describe how the shape of the energy eigenstates' (in both position and momentum space) associated probability densities change as n increases (t = 0). Explain why these densities should change in this way as n increases. Based on this answer, describe and sketch the n = 2 and n = 3 probability densities at t = 0 (which are not depicted).

5. The expectation value of x, <*x*>, is essentially an average of x weighted by the probability density in position space. The expectation value of p, <*p*>, is essentially an average of p weighted by the probability density in momentum space. What are <*x*> and <*p*> for t = 0 for these energy eigenstates?

6. Based on your answers to Questions 1 - 5, predict what will happen to the energy eigenstates as they evolve in time. Do this for both the position-space and momentum-space representations. Also predict how the probability densities will evolve in time.

7. Now watch the simulations again, this time watching them time evolve (press the Start button on the simulation). How do the energy eigenstates (in both position and momentum space) actually evolve in time? How do the probability densities evolve in time?

8. Does the probability density, <*x*>, or <*p*> change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?

9. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability density, and <*x*> and <*p*> to change over time for these superpositions? Why or why not?

Asymmetric Infinite Square Well Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical asymmetric infinite square well: a well with zero potential energy between $x = -L$ and $x = 0$, a constant potential energy, V_0, between $x = 0$ and $x = L$, and infinite walls at $x = -L$ and $x = L$. Here $L = 1$.

For Questions 1 - 5 load, but do not start, the simulations. In other words, do not press the Start button.

1. Load each state by double-clicking its green arrow. Describe and sketch how the shape of the energy eigenstates (in both position and momentum space) change as n increases ($t = 0$). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Explain why these states should change in this way as n increases. Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

2. Based on the answer to Question 1, describe and sketch the $n = 3$ and $n = 4$ energy eigenstates at $t = 0$ (which are not depicted).

3. Describe how the shape of the energy eigenstates' (in both position and momentum space) associated probability densities change as n increases ($t = 0$). Explain why these densities should change in this way as n increases. Based on this answer, describe and sketch the $n = 3$ and $n = 4$ probability densities at $t = 0$ (which are not depicted).

4. The expectation value of x, $<x>$, is essentially an average of x weighted by the probability density in position space. The expectation value of p, $<p>$, is essentially an average of p weighted by the probability density in momentum space. What are $<x>$ and $<p>$ for $t = 0$ for these energy eigenstates?

5. Based on your answers to Questions 1 - 4 predict what will happen to the energy eigenstates as they evolve in time. Do this for both the position-space and momentum-space representations. Also predict how the probability densities will evolve in time.

6. Now watch the simulations again, this time watching them time evolve (press the Start button on the simulation). How do the energy eigenstates (in both position and momentum space) actually evolve in time? How do the probability densities evolve in time?

7. Does the probability density, <*x*>, or <*p*> change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?

8. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability density, and <*x*> and <*p*> to change over time for these superpositions? Why or why not?

Ramped Infinite Square Well Exercises Questions:

These exercises refer to a particle trapped in a quantum-mechanical ramped infinite well: a well with potential energy $V(x) = \alpha x$ between $x = -L$ and $x = L$, and infinite walls at $x = -L$ and $x = L$. Here $L = 1$.

For Questions 1 - 5 load, but do not start, the simulations. In other words, do not press the Start button.

1. Load each state by double-clicking its green arrow. Describe and sketch how the shape of the energy eigenstates (in both position and momentum space) change as n increases ($t = 0$). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Explain why these states should change in this way as n increases. Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

2. Based on the answer to Question 1, describe and sketch the $n = 3$ and $n = 4$ energy eigenstates at $t = 0$ (which are not depicted).

3. Describe how the shape of the energy eigenstates' (in both position and momentum space) associated probability densities change as n increases ($t = 0$). Explain why these densities should change in this way as n increases. Based on this answer, describe and sketch the $n = 3$ and $n = 4$ probability densities at $t = 0$ (which are not depicted).

4. The expectation value of x, $\langle x \rangle$, is essentially an average of x weighted by the probability density in position space. The expectation value of p, $\langle p \rangle$, is essentially an average of p weighted by the probability density in momentum space. What are $\langle x \rangle$ and $\langle p \rangle$ for $t = 0$ for these energy eigenstates?

5. Based on your answers to Questions 1 - 4 predict what will happen to the energy eigenstates as they evolve in time. Do this for both the position-space and momentum-space representations. Also predict how the probability densities will evolve in time.

6. Now watch the simulations again, this time watching them time evolve (press the Start button on the simulation). How do the energy eigenstates (in both position and momentum space) actually evolve in time? How do the probability densities evolve in time?

7. Does the probability density, <*x*>, or <*p*> change over time for the energy eigenstates? Energy eigenstates are often called stationary states. Why? What is stationary?

8. If instead of an energy eigenstate, we had a system in a two-state superposition (both position and momentum space). Would you expect the wave functions, probability density, and <*x*> and <*p*> to change over time for these superpositions? Why or why not?

Followup I Exercises Questions:

These exercises refer to a particle trapped in an unknown quantum-mechanical well.

1. Begin by double-clicking the **State Shape Followup I** button. Describe and sketch the shape of the energy eigenstate ($t = 0$). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

2. Sketch the potential energy function that results in this particular energy eigenstate. Mark on your sketch your estimate for the energy eigenvalue of the eigenstate.

3. Compare and contrast this potential energy function with the ones from the exercises.

4. What energy eigenstate is it? Why? Sketch the adjacent energy eigenstates. In other words if $n = 2$ (the first-excited state) was depicted, you would sketch the $n = 1$ (ground state) and $n = 3$ (second-excited) energy eigenstates.

Followup II Exercises Questions:

These exercises refer to a particle trapped in an unknown quantum-mechanical well.

1. Begin by double-clicking the **State Shape Followup II** button. Describe and sketch the shape of the energy eigenstate (t = 0). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

2. Sketch the potential energy function that results in this particular energy eigenstate. Mark on your sketch your estimate for the energy eigenvalue of the eigenstate.

3. Compare and contrast this potential energy function with the ones from the exercises.

4. What energy eigenstate is it? Why? Sketch the adjacent energy eigenstates. In other words if n = 2 (the first-excited state) was depicted, you would sketch the n = 1 (ground state) and n = 3 (second-excited) energy eigenstates.

Followup III Exercises Questions:

These exercises refer to a particle trapped in an unknown quantum-mechanical well.

1. Begin by double-clicking the **State Shape Followup III** button. Describe and sketch the shape of the energy eigenstate (t = 0). For the position-space representation of the energy eigenstates, red/blue corresponds to what part of the state? Make sure to point out the regions in which the eigenstate is oscillatory, where it is not, and why this occurs for this well.

2. Sketch the potential energy function that results in this particular energy eigenstate. Mark on your sketch your estimate for the energy eigenvalue of the eigenstate.

3. Compare and contrast this potential energy function with the ones from the exercises.

4. What energy eigenstate is it? Why? Sketch the adjacent energy eigenstates. In other words if n = 2 (the first-excited state) was depicted, you would sketch the n = 1 (ground state) and n = 3 (second-excited) energy eigenstates.