AAPT Summer 2007 (Davidson and Greensboro)

Visualizing Tensors

Rob Salgado
Department of Physics, Syracuse University
physics.syr.edu/~salgado

All vectors are NOT created equal.

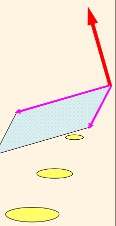
These directed quantities

- *displacements*
- *gradients*
- *"normals" to surfaces*
- *fluxes*

can be all be thought of as "vectors" only due to symmetries from additional geometrical structure

- *dimensionality of the vector space*
- *orientability of the vector space*
- *existence of a "volume-form"*
- *existence of a "metric tensor"*
- *signature of the metric*

which we can't always take for granted.



These symmetries blur the true nature of the directed quantity.

Only in three dimensions can you associate a "vector" with a parallelogram's area and orientation.

What is a vector?

"something with a magnitude and direction"?
Well... no... that's a "Euclidean Vector"
*(a vector with a particular **metric** [a rule for giving the lengths of vectors and the angles between vectors])*
Not all vectors in physics are Euclidean vectors.

*A **vector space** is a set with the properties of*
addition (the sum of two vectors is a vector)
scalar multiplication (the product of a scalar and a vector is a vector)
*Elements of this set are called **vectors**.*

What is a tensor?

*A **tensor** [of rank n] is a generalized type of vector [satisfying the above rules] that is a **multi-linear function of n vectors** (which, upon inputting n vectors, produces a scalar).*

They are useful for describing anisotropic (direction-dependent) physical quantities. For example,

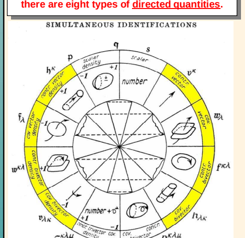
electromagnetic field tensor	*moment of inertia tensor*
stress tensor	*elasticity tensor*
riemann curvature tensor	*conductivity tensor*
metric tensor	

If the vector has, for example, 3 components, then a rank-n tensor has 3^n components.
(If you think about a vector as a column matrix, a tensor can be thought of as a [generalized] matrix.
But that's not really a good way to think about them...
although it might be a good way to calculate with them.)

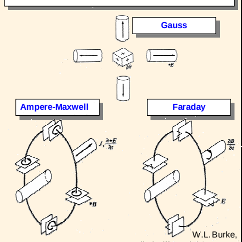
Some Motivations from the Literature

In three dimensions, there are eight types of directed quantities.

MAXWELL EQUATIONS FOR ELECTROMAGNETISM



J.A. Schouten, *Tensor Calculus for Physicists*



W.L. Burke, *Applied Differential Geometry*

Can we gain some physical and geometrical intuition by visualizing the natural form of these directed-quantities?

more References

Bamberg, P. and Sternberg, S. (1991) *A course in mathematics for students of physics.* (Cambridge University Press, Cambridge, England)
Burke, W.L. (1980) *Spacetime, Geometry, Cosmology.* (University Science Books, Mill Valley, California)
(1983) "Manifestly parity invariant electromagnetic theory and twisted tensors", *J. Math. Phys.* 24(1), January 1983, pp.65-69
(1985) *Applied Differential Geometry.* (Cambridge University Press, Cambridge, England)
Ingarden, R.S. and Jamiolkowski, A. (1985) *Classical Electrodynamics.* (Elsevier, Amsterdam)
Jancewicz, B. (1992) "Directed Quantities in Physics: Part I. Quantities Replacing Vectors" (Institute for Theoretical Physics (U. Wroclaw) preprint)
Misner, C.W. , Thorne, K.S., Wheeler, J.A. (1973) *Gravitation.* (W.H. Freeman, New York)
Schouten, J.A. (1924, 1954) *Ricci Calculus.* (Springer Verlag., New York)
(1951) *Tensor Analysis for Physicists.* (Dover Publ., New York)
Schouten, J.A. and Van Dantzig, D. (1939) "On ordinary quantities and W-quantities" *Compositio Mathematica* 7 , pp.447-473
Van Dantzig, D. (1934) "The fundamental equations of electromagnetism, independent of metrical geometry" *Proc. Cambridge Philosophical Society* 30 , pp.421-427
(1934) "Electromagnetism independent of metrical geometry 1. The foundations" *Akad. Wetensch. Amsterdam* 37 , pp.521-525
(1934) "Electromagnetism independent of metrical geometry 2. Variational principles and further generalizations of the theory" *Akad. Wetensch. Amsterdam* 37 , pp.526-531
(1934) "Electromagnetism independent of metrical geometry 3. Mass and Motion" *Akad. Wetensch. Amsterdam* 37 , pp.643-652
(1934) "Electromagnetism independent of metrical geometry 4. Momentum and Energy; Waves" *Akad. Wetensch. Amsterdam* 37 , pp.825-836
(1954) "On the Geometrical Representation of Elementary Physical Objects and the Relations between Geometry and Physics" *Nieuw. Achief. voor Wiskunde (3)* 2 , pp.73-89

VECTORS V^a

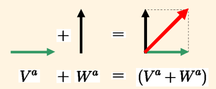
(a displacement)

Representations
- ordered PAIR OF POINTS with finite separation
- directed line-segment ("an ARROW")

The separation is proportional to its size.

(irrelevant features: thickness of the stem, size of the arrowhead)

Examples:
- displacement [in meters] as in $U=\frac{1}{2}k_{ab}r^a r^b$
- electric dipole moment $p^a=qd^a$ [in Coulomb-meters] as in $U=-p^a E_a$
- velocity v^a [in meters/sec] as in $K=\frac{1}{2}m_{ab}v^a v^b$
- acceleration a^a [in meters/sec²] as in $F_a=m_{ab}a^b$



(via the parallelogram rule)

COVECTORS (1-forms) ω_a

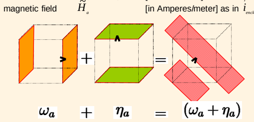
(a pair of neighboring equipotential surfaces)

Representations
- ordered PAIR OF PLANES ($\omega_a V^a = 0$ and $\omega_a V^a = 1$) with finite separation
- ("TWIN-BLADES")

The separation is *inversely*-proportional to its size.

(irrelevant features: size, shape, and alignment of the planar surfaces)

Examples:
- gradient $\nabla_a f$ [in [[f]] meters⁻¹]
- conservative force $F_a=-\nabla_a U$ [in Joules/meter]
- linear momentum "$p_a=\frac{\hbar}{\lambda^a}$" $=\hbar k_a$ [in action/meter] $p_a=\frac{\partial S}{\partial q^a}=\frac{\partial L}{\partial \dot{q}^a}$ $\quad p_a=-\frac{\partial H}{\partial q^a}=F_a$
- electrostatic field $E_a=-\nabla_a\phi$ [in Volts/meter] as in $\phi=-\int_\gamma E_a$
- magnetic field $\tilde{H}_a$ [in Amperes/meter] as in $i_{enclosed}=\oint_{\partial A}\tilde{H}_a$



(via the co-parallelogram rule)

BIVECTORS A^{ab}

(an area)

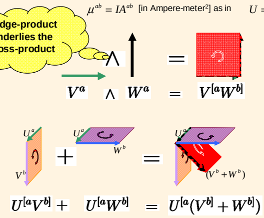
Representations
- ordered PAIR OF VECTORS (via the wedge product)
- directed two-dimensional planar region ("an AREA")

The area is proportional to its size.

(irrelevant features: shape of the planar surface)

Examples:
- area A^{ab} [in meters²] as in $A^{ab}=l^{[a}w^{b]}$
- force-couple (zero net-force "moment") [in (Newton/meter)-meter²] $M^{ab}=r^{[a}F^{b]}$
- magnetic dipole moment $\mu^{ab}=IA^{ab}$ [in Ampere-meter²] as in $U=-\mu^{ab}B_{ab}$

wedge-product underlies the cross-product



TWO-FORMS β_{ab}

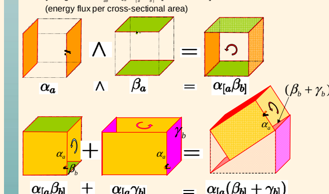
(a flux-tube: "density of field lines")

Representations
- ordered PAIR OF ORIENTED LOOPS
- an oriented cylinder ("a TUBE") with finite cross-sectional area

The cross-sectional area is *inversely*-proportional to its size.

(irrelevant features: shape of cross-section, length of the tube)

Examples:
- magnetic induction B_{ab} [Weber/meter²=Tesla] (magnetic flux per cross-sectional area) as in $\oiint_{\partial V} B_{ab}=0$
- electric induction $\tilde{D}_{ab}$ [Coulomb/meter²] (electric flux per cross-sectional area) as in $\oiint_{\partial V}\tilde{D}_{ab}=4\pi q_{enclosed}$
- current density $\tilde{j}_{ab}$ [Ampere/meter²] (charge flux per cross-sectional area) as in $\oint_{\partial A}\tilde{H}_a=\frac{\partial}{\partial t}\iint_A \tilde{D}_{bc}+4\pi\iint_A \tilde{j}_{bc}$
- Poynting vector $\tilde{S}_{ab}=\frac{1}{4\pi}E_{[a}\tilde{H}_{b]}$ [Watt/meter²] (energy flux per cross-sectional area)



In three dimensional space, the following are not directed-quantities.

TRIVECTORS V^{abc}

(a volume)

Representations
- ordered TRIPLE OF VECTORS
- sensed region ("a VOLUME") with finite size

The volume is proportional to its size.

(irrelevant features: shape of the volume)

Examples:
- volume V^{abc} [in meters³] as in $V^{abc}=l^{[a}w^b h^{c]}$

THREE-FORMS γ_{abc}

(a [volume] density)

Representations
- ordered TRIPLE OF COVECTORS
- oriented cell ("a BOX") with finite volume

The enclosed-volume is *inversely*-proportional to its size.

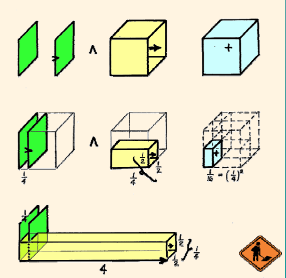
(irrelevant features: shape of the volume)

Examples:
- charge density $\tilde{\rho}_{abc}$ [in Coulombs/meters³] as in $q=\iiint_V \tilde{\rho}_{abc}$
- energy density $\tilde{u}_{abc}$ [in Joules/meters³] as in $\tilde{u}_{abc}=\frac{1}{8\pi}E_{[a}\tilde{D}_{bc]}$

VOLUME-FORM ε_{abc}

(in three dimensions)

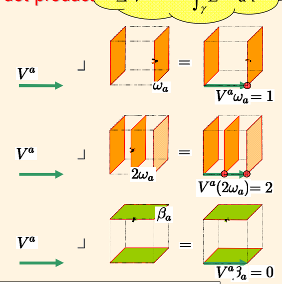
Specifying a volume form provides a rule to identify a vector with a unique two-form (in three dimensions), and vice versa. Vectors that are obtained from [ordinary] two-forms in this way are known as pseudovectors. (When a metric tensor is also specified, one can define additional identifications, called HODGE-DUALITY or the star-operation *.)



TRANSVECTION / INNER PRODUCT

(non-metrical "dot product")

underlies "potential difference" $\Delta V = -\int_\gamma \vec{E}\cdot d\vec{l}$



In *Gravitation* (Misner-Thorne-Wheeler), **this operation is described as counting the "bongs of a bell".**

METRIC TENSOR g_{ab}

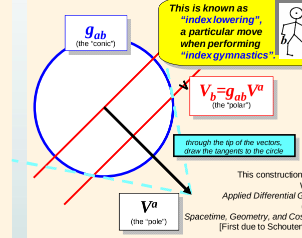
(ellipsoid of unit-vectors)

Representations
- depending on the signature: certain cross-sections may be ellipses, hyperbolas, or parallel lines [for degenerate cases]

The figure defines the set of unit-vectors.

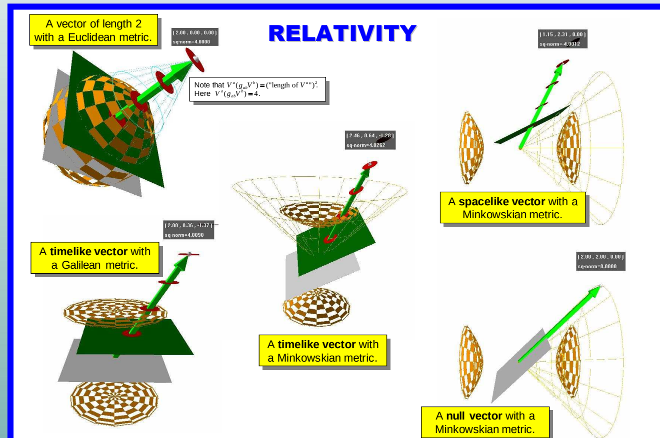
A metric tensor is a symmetric tensor used to
- **assign a "magnitude" to a vector** $\|V\|^2=g_{ab}V^aV^b$
- **assign an "angle" between vectors.**
- **identify a vector with a unique covector,** $V_b=g_{ab}V^a$ **called its "[metric-]dual"**

This is known as "index lowering", a particular move when performing "index gymnastics".

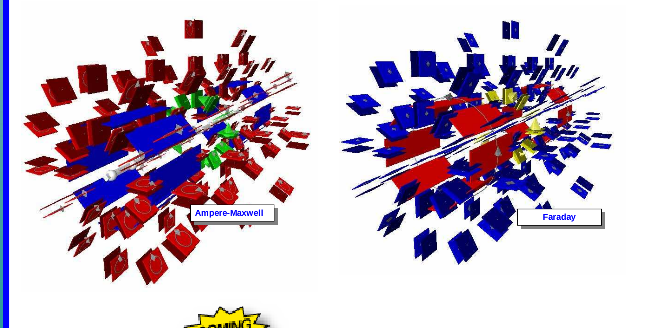


This construction is due to W. Burke, *Applied Differential Geometry*. (See also *Spacetime, Geometry, and Cosmology*.) [First due to Schouten (1923)?]

RELATIVITY



ELECTROMAGNETISM



COMING SOON

a VPython module and a Maple package to perform and visualize calculations in tensor algebra